Induced Characters

G - finite group

$H \leq G$, $\qquad m = [G:H]$

$$\mathrm{Res}_H^G : L^c(G) \longrightarrow L^c(H)$$
$$\quad f \longmapsto f' := \mathrm{Res}_H^G(f)$$

restriction $\qquad f'(h) := f(h) \quad \forall\, h \in H$

Prop: If χ = character of a G-rep, then

$\chi' = \mathrm{Res}_H^G(\chi)$ is a character of an H-rep

Proof:

$$\begin{tikzcd} H \arrow[r, hook, "\mathrm{ind}"] \arrow[rr, bend left, "\varphi'"] & G \arrow[r, "\varphi"] & GL(V) \end{tikzcd}$$

$$\chi_{\varphi'} = \mathrm{Res}_H^G(\chi_\varphi)$$

$$\mathrm{Ind}_H^G : L^c(H) \longrightarrow L^c(G)$$

Induction

$$f \longmapsto f' = \mathrm{Ind}_H^G f$$

$$f'(g) := \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} f(xgx^{-1}) \qquad g \in G$$

$$= \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}} f(x_i g x_i^{-1})$$

$R = \{x_i\}$ = set of representatives of right H-cosets

i.e, $x \in G$, $Hx = Hx_i$ for exactly one $x_i \in R$

$$\mathrm{Ind}_H^G : \underline{L^c(H)} \rightleftarrows \underline{L^c(G)} : \mathrm{Res}_H^G$$

"adjoint"

Prop: $f \in L^c(H)$, $f' \in L^c(G)$

$$\langle f, \mathrm{Res}_H^G(f') \rangle_H = \langle \mathrm{Ind}_H^G(f), f' \rangle_G$$

Prop: If χ = character of an H-rep, then $\chi' = \mathrm{Ind}_H^G(\chi)$ is a character of a G-rep

Note: If φ H-rep, $\chi = \chi_\varphi$

If ψ G-rep, $\chi' = \chi_\psi = \mathrm{Ind}_H^G(\chi)$

then
$$\begin{array}{ccc} \dim \psi & = & [G:H] \cdot \dim \varphi \\ \| & & \| \\ \chi'(e) & & [G:H]\, \chi(e) \end{array}$$

Example: $S_3 \simeq H \leq G = S_4$

$$H = \{g \in G \mid g(4) = 4\}$$

irreducible character

S_3	e	(12)	(123)
χ	2	0	-1

$\Downarrow$

$$\boxed{\chi'(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \chi(xgx^{-1})}$$

S_4	e	(12)	$(12)(34)$	(1234)	(123)
χ'	8	0	0	0	-1

$$\langle \chi', \chi' \rangle_{S_4} = 3 = 1^2 + 1^2 + 1^2$$